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THEORY OF ELASTIC-PLASTIC DEFORMATION AND ITS APPLICATIONS

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/Figures referred to herein are appended./

Theory

Only the basic properties of metals are investigated; namely, elasticity and the nonlinear dependence of stress upon deformation beyond the elastic limit and also the nature of "load-relieving" and repeated load (Figure 1).

For active plastic deformation

$$\sigma_A = \Phi(e_A), \qquad e_A = e_{pA} + \frac{\sigma_A}{E} \qquad (1)$$

For load-relieving

$$\sigma = E(e - e_{pA}), \qquad d\sigma = E\,de, \qquad \sigma - \sigma_A = E(e - e_A). \qquad (2)$$

The complex strained state of an element of a solid is characterized by the uniform universal tension σ (pressure is $-\sigma$) and by the stresses of form variation $S_{\alpha\beta}$ ($\alpha, \beta = x, y, z$), comprising the deviator of stress (Figure 2); the sum of its main components is equal to zero:

$$\sum_{\alpha = x, y, z} S_{\alpha\alpha} = 0,$$

and therefore it is related only to the distortion of the form, and not to any change in volume of an element.

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The deformed state of an element is determined by the universal extension (or compression) e, equal to one third of the relative change in volume and deformations of form variation $\text{Э}_{\alpha\beta}$ ($\alpha, \beta = x, y, z$), which comprise the deformation deviator:

$$\sum_{\alpha = x, y, z} \text{Э}_{\alpha\alpha} = 0.$$

The second invariant of the stress deviator is called the intensity of the tangential stresses or the octahedral stress

$$\tau_i = \frac{1}{\sqrt{3}} \sqrt{S^2_{\alpha\beta}} = \frac{1}{3}\sqrt{\Sigma S^2_{\alpha\beta}} \qquad \sigma_i = \frac{3}{\sqrt{2}}\tau_i \quad (3)$$

The quantity σ_i is called the intensity of stress or the reduced stress; in the case of simple tension (elongation) of the specimen the quantity σ_i is equal to the tensile stress.

If at any point of the solid an octahedron is mentally cut out, the axes of which coincide with the main axes of stress, then only tangential stress τ_i and universal uniform tensile stress σ act on its faces (Figure 3). The second invariant of the deformation deviator is called the octahedral deformation:

$$\gamma_i = \frac{2}{\sqrt{3}} \sqrt{\text{Э}^2_{\alpha\beta}} = \frac{2}{\sqrt{3}} \sqrt{\Sigma \text{Э}^2_{\alpha\beta}} \qquad e_i = \frac{1}{\sqrt{2}}\gamma_i \quad (4)$$

The quantity e_i is called the intensity of deformation or the reduced elongation.

In solids of complex form, for a complex nature of load, the strained and deformed state (in general the mechanical state) is determined by the quantities σ, e, $S_{\alpha\beta}$, $\text{Э}_{\alpha\beta}$ and by their variations during the whole period of deformation. Therefore, it is possible to examine the very general relation

$$A_1 S_{\alpha\beta} - B_1 \text{Э}_{\alpha\beta} + A_2 \frac{\partial S_{\alpha\beta}}{\partial t} - B_2 \cdot \frac{\partial \text{Э}_{\alpha\beta}}{\partial t} + \dots + \int A_1' S_{\alpha\beta}\, dt - \int B_1' \text{Э}_{\alpha\beta}\, dt + \dots = 0, \quad (5)$$

in which A_n, B_n, A_n' B_n' are any functions of time, invariants $S_{\alpha\beta}$ and $\text{Э}_{\alpha\beta}$, their derivatives, σ_i, e_i etc.

Equation (5) is the mathematical expression for all the known theories of continuous deformable solids, as well as for an infinite number of still unexamined new theories of continuous deformable solids. It includes principles of hydrodynamics, theories of elasticity, theories of free-flowing material, all known theories of plasticity, theories of "heredity," basic theories of creep, and others. It all depends upon the number of terms to which we limit ourselves in equation (5) and upon the manner of choosing the coefficients. It is the most general integrodifferential tensor-linear equation relative to tensors $S_{\alpha\beta}$ and $\text{Э}_{\alpha\beta}$; therefore, it encompasses, so to speak, the whole past and future of any element of a solid. Of course, an even more general equation can be written, but for our purposes this is not necessary.

We call the variation simple if, during the period covering the complex strained state of the element (called simple load), the deviators

$$\frac{S_{\alpha\beta}}{\sqrt{S^2_{\alpha\beta}}} \quad \text{and} \quad \frac{\text{Э}_{\alpha\beta}}{\sqrt{\text{Э}^2_{\alpha\beta}}} \quad (6)$$

which are called the "directional" tensors (because they determine the main axes of stress and strain) do not change during this period.

Theorem: For simple load, equation (5) can be transformed into a binomial equation; for example, in the form

$$S_{\alpha\beta} = C\text{Э}_{\alpha\beta} \tag{7}$$

or in any form containing a finite number of terms.

For example

$$1)\ S_{\alpha\beta} = C_1 \frac{\partial \text{Э}_{\alpha\beta}}{\partial t},\quad 2)\ \frac{\partial \text{Э}_{\alpha\beta}}{\partial t} = \frac{1}{2G}\frac{\partial S_{\alpha\beta}}{\partial t} + C_2 S_{\alpha\beta}$$

$$3)\ 2G\frac{\partial \text{Э}_{\alpha\beta}}{\partial t} = \frac{\partial S_{\alpha\beta}}{\partial t} + g(\sigma_i)\frac{\partial \sigma_i}{\partial \lambda} S_{\alpha\beta} \tag{8}$$

The first of the forms (8) is applied in the hydrodynamics of viscous liquids and in the Sen-Venan-Levi-Mises theory of plasticity; the second encompasses the Prandtl-Race theory of plasticity, the Maxwellian theory of relaxation, and others; the third represents one of Prager's recent theories of plasticity. It is possible to obtain a form of Belyayev's theory of plasticity, and Bingemov's and our form of the theory of visco-plastic solids, and others.

Thus, in the case of simple stress all basic theories of plasticity agree with each other. We select form (7), representing a generalized form of the Hank-Mises theory of plasticity, as the most simple to write, although in the case of simple load, they are all equivalent and represent a single theory of plasticity, as developed in our works.

Investigation of all existing experimental material has shown that in the case of simple load, the equality of the deviators of stresses and strains, their rates, etc., is fulfilled to a satisfactory degree of accuracy.

If the influence of rate, time, etc., is disregarded, or, rather, if the tests on the complex strained state in the sense of temporary conditions are conducted in the same way as ordinary tensile tests, then the diagram of σ versus σ_i agrees with the diagram of simple tension, that is:

$$\sigma_i = \Phi(e_i). \tag{9}$$

Volumetric strain satisfied Hooke's law

$$\sigma = 3ke \tag{10}$$

Equation (5) for a simple load on an element of the solid leads to the following relationship between stress and strain:

$$S_{\alpha\beta} = \frac{2\sigma_i}{3e_i}\text{Э}_{\alpha\beta},\qquad \alpha,\beta = x, y, z. \tag{11}$$

Formulas (9), (10), and (11) also represent the law of active plastic deformation. Hooke's law is obtained as a special case when $\sigma_i = 3Ge_i$.

Elastic deformation during "load-relieving" of an element of the solid conforms to the law of "load-relieving":

$$S_{\alpha\beta} = 2G(\text{Э}_{\alpha\beta} - \text{Э}^{(p)}_{\alpha\beta}),\quad \alpha,\beta = x,y,z,\quad \sigma = 3ke \tag{12}$$

for which $\text{Э}^{(p)}_{\alpha\beta}$ is the component of the plastic deformation corresponding to stress $S_{\alpha\beta}$.

The following question arises: Can there actually be instances where a solid of complex form for a complex nature of the load is so deformed that every one of its elements experiences a simple load stress insofar as a single theory of plasticity exists in this case. The answer to it is the following hypothesis about simple load stress: If the loads of any complex form, applied to a solid of arbitrary form, vary with time in a similar manner (that is, in proportion to one general parameter), then simple load stress occurs at every point of the solid. Thus a single theory of plasticity encompasses a wide range of technical problems.

The hypothesis of simple load stress is a theorem that has been demonstrated for materials whose volumetric variation during deformation can be disregarded and whose law (9) σ_i versus e_i is rather closely approximated by power-series functions.

The equations of equilibrium of an element of the solid have the form:

$$\frac{\partial \sigma}{\partial \beta} + \frac{\partial S_{x\beta}}{\partial x} + \frac{\partial S_{y\beta}}{\partial y} + \frac{\partial S_{z\beta}}{\partial z} + \rho F_{\beta} = 0, \quad \beta = x, y, z \qquad (13)$$

where F_{β} is the volumetric force.

The values of σ and $S_{\alpha\beta}$, in accordance with (9), (10), and (11) can be assumed to be already substituted in these equations; consequently, there are three equations containing three unknown components of translation of any point. For any material possessing hardening, the equations are of the elliptical type; that is, similar to the equations in the theory of elasticity. For ordinary boundary conditions, their solution is unique; and the theorem of minimum work of internal forces and others also hold true.

One of the effective methods of solving problems of plasticity is shown; namely, the method of elastic solutions. It consists of the fact that the formula is written in the form

$$\sigma_i = 3Ge_i[1 - \omega(e_i)], \qquad 0 < \omega < 1 \qquad (14)$$

and in the first approximation it is assumed that $\omega = 0$. A problem of elasticity is obtained, which is solved, and translations and deformations are determined at all points of the solid. The distribution of stress in the first approximation is found in accordance with the formulas of the theory of plasticity (9), (10), (11). Further, the function ω is found in the first approximation and the problem of the theory of elasticity is again found from equation 13, which corrected mass forces; its solution gives the second approximation, etc. In certain cases and most often in problems on stress concentration, the first approximation gives in practice a satisfactory picture (recently again confirmed by G. V. Uzhik in the Metallurgical Institute of the Academy of Sciences USSR).

We go on to some basic applications of the theory of plasticity.

Some of the Simplest Problems

1. The theory developed here in the greatest detail is that of the bending of a beam by transverse forces. As an example, let us examine a beam of rectangular cross section of width l and height h.

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In a certain cross section, let M be the bending moment; I, the moment of inertia; K curvature σ maximum flow of the material; $e_s = \frac{\sigma}{E}$ corresponding deformation. If the diagram σ_i versus e_i is represented by a broken line so that

$$\omega = 0, \quad e_i < e_s, \quad \omega = \lambda\left(1 - \frac{e_s}{e_i}\right), \quad e_i > e_s, \tag{15}$$

$$\lambda = 1 - \frac{1}{E}\frac{d\sigma_i}{de_i}$$

then the equation of the bent axis of the beam will be

$$K - \lambda\left(K - \frac{3}{2} + \frac{\lambda}{2K^2}\right) = K_0, \tag{16}$$

where k_0 is the reduced moment and k is the reduced curvature:

$$K_0 = \frac{Mh}{2\sigma_s I}, \qquad K = \frac{h}{2e_s}\chi. \tag{17}$$

The residual curvature of the beam after removal of the load is:

$$\bar{K} = K - K_0.$$

Figure 4 shows the general and residual stresses.

2. The stability of a compressed bar has long been investigated. During the loss of stability in the cross section, a region of load-relieving appears, placed at a distance y_0 from the center plane of the beam ($y_0 < 0$). The critical force can be found by Euler's corresponding formula, if instead of Young's modulus E we take Karman's modulus; that is, if we multiply the first by

$$K_r = 1 - \left(1 - \frac{1}{E}\frac{d\sigma_i}{de_i}\right)\frac{I_0 - y_0 S_0}{I}, \tag{18}$$

where I_0, S_0 respectively are the moment of inertia and the static moment of part of the cross section, in which, during loss of stability, an active plastic deformation takes place. For a rectangular cross section we have

$$K_r = \frac{4\frac{d\sigma_i}{de_i}}{\left(\sqrt{E} + \sqrt{\frac{d\sigma_i}{de_i}}\right)^2}$$

The tangent of the angle of slope of the curve σ_i versus e_i, equal to $\left(\frac{d\sigma_i}{de_i}\right)$ on the diagram describing the tension (elongation) of the specimen is called the modulus of hardening.

3. During strong twisting of a roller, a region of plastic deformation appears and extends up to the external surface (Figure 5). The dependence of the angle of torsion M upon twist τ is defined by the formula following:

$$M = GI_p\tau - \frac{\lambda G}{\tau^3}\left(I_p\tau^4 - e_s S_p\tau^3\sqrt{3} + \frac{3}{4}\pi e_s^4\right) \tag{19}$$

where G is the shear modulus and I_p and S_p are respectively the polar and static moments of inertia of the section

$$I_p = \frac{\pi a^4}{2} \qquad \text{and} \qquad S_p = \frac{2\pi a^3}{3}$$

When the moment is reduced to zero, the residual twist is:

$$\tilde{\tau} = \tau - \frac{M}{GI_p}$$

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during which the repeated moments of the previous direction produce only an elastic twisting, if they do not exceed in magnitude the first twisting moment which produced plastic deformation.

4. Stresses and strains of a hemisphere and tube under the influence of an external pressure P_b and internal pressure P_a are found, within the elastic limits by means of Lame's formulas. In order that the whole mass of the hemisphere should pass into the plastic state, the necessary pressure is:

$$|P_a - P_b| \geqslant \left[\frac{3\lambda K \ln\frac{b}{a}}{2\left[G(1-\lambda)+\frac{3}{4}K\right]} + \frac{2(1-\lambda)\left(G+\frac{3}{4}K\right)(b^3-a^3)}{3\left[G(1-\lambda)+\frac{3}{4}K\right]a^3}\right]\sigma_s \quad (20)$$

and the increment of the inner radius is:

$$w_a = \frac{ab^3}{4G(1-\lambda)(b^3-a^3)}\left[P_a - P_b - 2\lambda\sigma_s \ln\frac{b}{a}\,\mathrm{sign}\,w\right] + \quad (21)$$
$$+ \frac{(a^3P_a - b^3P_b)a}{3K(b^3-a^3)}$$

Here k is the modulus of volumetric compression.

For $\frac{b}{a} = 2$, $\lambda = 0.95$, $P_b = 0$, $P_a = 1.7\,\sigma_s$ and $K = 2G$

we have:

$$\frac{w_a}{a} = 0.01092 - 0.00020$$

The second term reflects the influence of the compressibility of the metal. It is 55 times smaller than the first term; therefore, the influence of compressibility during plastic deformation is small.

If in the tube under pressure a plastic region of radius $a \leqslant r^* \leqslant b$, appears, then for decreasing pressures a residual deformation will appear in the tube, defined by the graph in Figure 7.

For thick-walled tubes $\alpha = \frac{a}{b} < 0.5$, the residual deformations are not large. For example, if the thickness of the wall, by halves or less, goes beyond the elastic limit ($r^* < 1.5\,a$), the residual deformations will be up to 10 percent of the usual strains.

For simultaneous actions of the axial force $\underline{P}$ and pressures P_a and P_b, the tube goes beyond the elastic limit if the values:

$$p' = \frac{(P_a - P_b)\sqrt{3}}{\sigma_s}, \quad p'' = \frac{P - \pi(a^2P_a - b^2P_b)}{\pi a^2\sigma_s}$$

satisfy the equation:

$$p' = \ln\left[\frac{-p''^2 + \beta^4 - 1 + \sqrt{(-p''^2+\beta^4-1)^2 - 4p''^2} \text{ [illegible]}}{\alpha^2(+p'^2+\beta^4-1) + \sqrt{\alpha^4(+p''^2+\beta^4-1)^2} \text{ [illegible]}}\right] \quad (22)$$

Similar relations determining the supporting capacity of solids are called "ultimate relations." If the tube does not elongate along the axis, then from (9) we find:

$$p'' = 0, \quad (P_a - P_b) = \frac{2\sigma_s}{\sqrt{3}}\ln\frac{b}{a}$$

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Equilibrium in Plates and Shells

The internal forces and moments arising in shells during elastic-plastic deformations are expressed as strains and torsions (twisting) in the mean surface by rather complicated formulas. Figure 8 shows the forces acting upon an element of a shell. Inverse functional relations are also found only if the material possesses hardening. There exists a potential U of the generalized forces such that:

$$T_1 = \frac{\partial U}{\partial e_1}, \qquad T_2 = \frac{\partial U}{\partial e_2}, \qquad T_{12} = \frac{1}{2}\frac{\partial U}{\partial e_{12}},$$

$$M_1 = -\frac{\partial U}{\partial \kappa_1}, \qquad M_2 = -\frac{\partial U}{\partial \kappa_2}, \qquad M_{12} = -\frac{1}{2}\frac{\partial U}{\partial \kappa_{12}} \tag{23}$$

in which U represents the work of the internal forces, which is equal to the unit area of the mean surface.

If the material of the shell does not possess hardening, then one "ultimate relation" exists between the forces and moments similar to Mises' condition for plasticity. It has approximately the form:

$$\frac{1}{T_s^2}\left(T_1^2 - T_1T_2 + T_2^2 + 3T_{12}^2\right) + \frac{1}{M_s^2}\left(M_1^2 - M_1M_2 + M_2^2 + 3M_{12}^2\right) + \tag{24}$$

$$+ \frac{1}{T_sM_s\sqrt{3}}\left(T_1M_1 - \frac{1}{2}T_1M_2 - \frac{1}{2}T_2M_1 + T_2M_2 + 3T_{12}M_{12}\right) = 1.$$

Here

$$T_s = \sigma_s h, \qquad M_s = \frac{\sigma_s h^2}{4}$$

and h is the thickness of the shell. "The ultimate relation" (11) permits one to find the supporting capacity of the shells.

1. Deformation of Plates and Their Planes

If the main stresses σ_1 and σ_2 have the same sign, then the equations of the plane problem will be of the elliptical type. For materials not possessing hardening, the condition for plasticity τ_{max} = const which almost agrees with Mises' condition, gives either $\sigma_2 = \pm 2K$ or $\sigma_2 = 2K$:

$$\sigma_s \leqslant 2K \leqslant \frac{2\sigma_s}{\sqrt{3}},$$

and therefore, one of the family of lines of main stresses consists of straight lines (Figure 9) and the other, of parallel curves orthogonal to the straight lines. This circumstance permits one to solve problems of supporting capacity, a part of which are shown in Figure 10.

In the first case, the supporting capacity is determined by formula:

$$\sigma_1 = 2K;$$

in the second, by pressure:

$$p = 2K\frac{b-a}{a},$$

in the third, by

$$P = 2K\frac{H}{R},$$

where R is the radius of curvature of the outer contours and H is the width of the ring.

In the case where a gravitational force ρg (Figure 11) acts along the rectilinear family of trajectories and the stress p acts on the outer contour and the stress q, on the inner, then the supporting capacity is defined by formula:

$$P - q\frac{R''}{R'} + \frac{1}{R'}\int_{R''}^{R'} \rho g R\,dR = 2K\frac{H}{R} \quad (25)$$

Hence, in particular, for a round disc ($R' = b$, $R'' = a$) rotating with angular velocity ω:

$$P - q\frac{a}{b} + \frac{\rho\omega^2}{3b}(b^3 - a^3) = 2K\frac{b-a}{a}$$

2. Bending of Plates (Figure 12)

The problem of the bending of plates generally leads to a variational equation (calculus of variation):

$$\frac{1}{2}\iint I(P_x)\,\delta P_x\,dx\,dy = \iint q\,\delta w\,dx\,dy, \quad (26)$$

where w is the bend and q is the load and

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \varkappa_1 = -\frac{4}{3I}\left(M_1 - \frac{1}{2}M_2\right), \\ \frac{\partial^2 w}{\partial y^2} &= \varkappa_2 = -\frac{4}{3I}\left(M_2 - \frac{1}{2}M_1\right), \qquad (27) \\ \frac{\partial^2 w}{\partial x\,\partial y} &= \varkappa_{12} = -\frac{2}{I}M_{12}. \end{aligned}$$

The function I is determined by means of the diagram σ_i versus e_i, thus:

$$I = \frac{\sqrt{3}}{P_x^{\frac{3}{2}}}\int^{\frac{2}{\sqrt{3}}\sqrt{P_x}} \sigma_i e_i\,de_i,$$

$$P_x = \varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2 + \varkappa_{12}^2$$

The equation obtained from (26) can be integrated by the method of elastic solutions which method leads to a converging process of successive approximations. Equation 26, however, can also be solved by Ritz's method.

We can obtain a reliable approximate solution, if we use an elastic form of bending. Let us assume:

$$w = c\frac{a^4}{D}\vec{w}(x, y), \quad (28)$$

where a and D are respectively a characteristic dimension and the hardness of the plate and c is an undetermined constant. Let us assume for $c = q_0$, where q_0 is a characteristic constant value of the load, that formula (28) gives an exact solution of the elastic problem. Then during plastic deformation we obtain

$$q_0 = C\left[1 - \frac{\iint \Omega\,\bar{P}_x\,dx\,dy}{\iint \bar{q}\,\bar{w}\,dx\,dy}\right] \quad (29)$$

where Ω is a definite function of p_x and $\bar{q} = \frac{q}{q_n}$.

It is easy to find exact solutions for the problems of deflection in round plates subjected to a symmetrical load. An example is shown in Figure 13.

The problem leads to the integration of an equation of the first order:

$$\frac{d\nu}{d\chi} = \frac{4(1-\Omega) - (3\chi - 2\nu)\frac{\partial\Omega}{\partial\chi} - \frac{\rho}{\nu}}{2(1-\Omega) + (3\chi - 2\nu)\frac{\partial\Omega}{\partial\nu}} = f(\nu, \chi), \qquad (30)$$

where $\chi = \frac{\kappa_2 h}{2e_s}$, $\nu = -r\frac{d\chi}{dr}$, and P is the intersecting force at $r = b$, and also:

$$\Omega = 0, \ (e \leq 1), \quad \Omega = \lambda\left(1 - \frac{3}{2e} + \frac{1}{2e^3}\right) \ (e \geq 1),$$

$$e = 2\sqrt{\chi^2 - \chi\nu + \frac{1}{3}\nu^2}, \quad \rho = \frac{bhP}{De_s}$$

If the outer and inner contours are free from moments ($M_2 = 0$), then

$$\begin{matrix} r = a \\ r = b \end{matrix} \quad 3r - 2\nu = 0, \quad e = \chi.$$

For a round plate subjected to a uniform load, an approximate solution by formula (29), for the case of a complete elastic-plastic state, gives:

$$\frac{21qa^2}{16\sigma_s h^2} = (1-\lambda)e_0 + 2.22\lambda - \frac{2.89\lambda}{e_0^2}, \quad e_0 = \frac{21qa^2}{16\sigma_s h^2} \qquad (31)$$

Comparing the approximate solution with the accurate solution, we discover a difference in deflection of about 1 percent and a difference in stresses of about 15 percent.

For a square plate, an approximate solution is taken in form (28) where

$$\bar{w} = \frac{64}{\pi^4}\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}.$$

Figures 14, 15, and 16 show the plates and the distribution of the intensity of strain (deformation); that is, also the maximum reduced stress.

Deflection in the center of the plate is:

$$w_0 = \frac{4}{\pi^2}\frac{a^2 e_s}{h}e_0,$$

where e_0 is connected with the reduced load

$$K = \frac{144qa^2}{\pi^4\sigma_s h^2}$$

by the following relation:

$$K = (1-\lambda)e_0 + 2.44\lambda - \frac{2.52\lambda}{e_0^2} \qquad (e_0 > 5).$$

In analyzing the relations between the force k and the shift in position e_0 of the solid, we see that the relations are analogous to those in diagram of σ_i versus e_i.

The supporting capacity of the plates, that is, the maximum value of the load for which they lose equilibrium if the material does not possess hardening, is found by a formula similar to Raleigh's formula for the frequency of oscillation of a membrane:

$$K = \frac{a^2}{2\sqrt{3}} \; \frac{\iint \delta\sqrt{p_1}\,dx\,dy}{\iint \bar{q}\,\delta\, w\,dx\,dy}, \qquad (32)$$

$$K = \frac{q_0 a^2}{\sigma_s h^2}, \quad q_0 \bar{q} = q(x, y).$$

The appropriate form of deflection w, in agreement with this formula, always gives a quite exact value of the maximum loads.

$$K = 4\frac{bP}{h^2\sigma_s} \qquad \left(1 \leq \delta \leq \frac{2}{\sqrt{3}}\right)$$

The exact solution for the problems on the supporting capacity of round plates (Figure 17) leads to the dependence (Figure 18) of the magnitude upon the ratio of the dimensions of the plates.

For a continuous circular plate subjected to a symmetrical load (Figure 19):

$$q = q_0 \bar{q}(r), \quad f(\rho) = \int_0^{\rho} \bar{q}\, \rho\, d\rho, \quad \rho = \frac{r}{a}$$

we have:

$$\frac{q_0 a^2}{\sigma_s h^2} = K = \frac{\delta}{4\int_0^1 f(\rho)\,d\rho}$$

A comparison of the exact calculations with the approximate calculations of the supporting capacity according to formula 32 gives in all cases analyzed by us a difference that fall in the limits of possible values of the quantity δ :

$$1 \leq \delta \leq \frac{2}{\sqrt{3}},$$

which corresponds to Mises' and Sen-Venan's conditions for plasticity.

3. Nonmoment Theory of Shells of Revolution

For small strains the problem of elastic-plastic deformations receives a simple general solution in those cases where the load varies smoothly in the surface of the shell. If, during transition from one section along the generatrix to another section for a distance of the order $\sqrt{hR}$, the load changes slightly (less than $\sqrt{\frac{h}{R}}$ in comparison with 1), then the state is nonmoment (Figure 20.)

New, interesting problems are obtained if one examines large deformations. Figure 21 shows a circular plate with a large deflection. Its exact solution for any functional relationship between σ_i and e_i expressed in power terms is given as a power series. In the case of a cubic parabola:

$$e_i = \left(\frac{\sigma_i}{\sigma}\right)^3$$

then the relation obtained between the deflection in the center and the pressure p is:

$$w_0 = 0.678\,a\,\sqrt[5]{\frac{a^3 p^3}{8h^3\sigma^{3}}}$$

An analysis shows that the surface obtained is almost spherical

Spherical and cylindrical shells, plastically deformed under the action of internal pressure, are considerably strengthened if their material is capable of cold-hardening. For a spherical shell (Figure 22) the relation between pressure and ultimate deformation is thus:

$$p = \frac{2\lambda h_0}{R_0} \sigma_s \frac{1 + m(\rho - 1)}{\rho^3},$$

$$m = \frac{2(1-\lambda)}{\lambda e_s}, \qquad \rho = \frac{R}{R_0}$$

Steel shells, for deformations of the order 5 - 7 percent, can possess hardening of the order 50 percent and above.

Rolling and drawing of tubes (Figure 23) are computed exactly. For example, Figures 24 and 25 show solutions for the case of a conical die.

Here $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are the meridional and tangential stresses reduced to maximum flow σ_s, φ is the cosine of half the angle of conicity, $\beta_s = \frac{h_0}{a} \sigma_s \delta$; δ is the relative thickness of the wall. The curves show the surface-pressure distribution, the variation in wall thickness, and the stress as functional dependencies upon the relative (reduced) radius $\bar{r} = \frac{r}{a}$.

Curves for $F(\bar{r})$ give the distribution of axial elongation and permit one to find the amount of contraction and elongation:

$$\frac{\sqrt{1-\rho^2}}{a} (L - L_0) = \int F(\bar{r})\, d\bar{r}$$

The moment theory of deflection of a cylindrical shell subjected to a symmetrical load (Figure 26) results in a nonlinear equation

$$\frac{d^4\bar{w}}{d\xi^4} + 4\bar{w} = \bar{P} + \lambda \delta t + \lambda \frac{d^2 \delta m}{dx^2},$$

into which enter the corrections in tangential force δ_t and moment δ_m because of plastic deformation. Three regions appear in the material: a clearly plastic region ($\pm \bar{x}_0$), elastic-plastic region $\bar{x}_1 - \bar{x}_0$, and an elastic.

Applying the method of elastic solutions, we solve the problem in A. N. Krylov's beam functions I_1, I_2, I_3, I_4:

$$\frac{dI_2}{d\xi} = I_1, \quad \frac{dI_3}{d\xi} = I_2, \quad \frac{dI_4}{d\xi} = I_3, \quad \frac{dI_1}{d\xi} = -4I_4$$

The third approximation differs almost imperceptibly from the second; and in conformity with the problem of cyclic pressure, a relation is obtained among the quantities $\bar{p}$, α, β, which are dependent upon the force P and the deflection under force w and upon the curvature:

$$\bar{p} = \frac{4ap}{h\sigma_s}, \qquad \alpha = \frac{w_0}{a e_s}.$$

The relationship is given in the form of graphs (Figure 27) for various values of the parameter λ, which characterizes the capacity of the material of the shell to harden.

The problem on the supporting capacity of a shell for various loads is exactly solved, insofar as the ultimate relation between forces and moments acquires a simple form:

$$\frac{T_1^2}{T_s^2} + \frac{3}{4} \frac{m_1^2}{M_s^2} = 1$$

and the problem becomes statically determinable. For concentrated cyclic pressure P (Figure 28), the maximum magnitude of this pressure is given by formula

$$P = \frac{3\sqrt{\pi}}{\sqrt{3\sqrt{5}}} \quad h\sigma_s\sqrt{\frac{2h}{3a}} \, .$$

This quantity is in very close agreement with the graph (Figure 28) for $\lambda = 1$, which again and in the most unfavorable case illustrates the accuracy of the method of elastic solutions for small successive approximations.

The Stability of Plates and Shells

The phenomenon of the loss of stability in shells beyond maximum elasticity is different from the phenomenon of the loss of stability in the elastic state in that it is accompanied by the elongations of the center surface and by the formation of zones of active and passive strains. All equations of stability known in the mechanics in the first approximation appear to be linear. Here there is a nonlinear relation between the infinitely small strains, stresses, forces, and moments; therefore, the obtained equations of stability are nonlinear.

The problem of stability in plates, for example, is solved with the help of two simultaneous equations relative to the deflecti[illegible]n and to the function of "auxiliary" stress F:

$$\nabla^4 w - \frac{h \varphi}{D} \chi = \frac{3}{4}\left(\frac{\partial^2}{\partial x^2}\bar{X}_x + \frac{\partial^2}{\partial y^2} Y_y + 2\frac{\partial^2}{\partial x \partial y} \bar{X}_y\right)\lambda\, \xi^2(3-2\xi)\,\chi,$$

$$\nabla^4 F = \frac{h}{2}\left(\frac{\partial^2}{\partial y^2}\bar{S}_x + \frac{\partial^2}{\partial x^2}\bar{S}_y - 3\frac{\partial^2}{\partial x \partial y}\bar{X}_y\right)\lambda\,\xi^2\chi.$$

In these equations (Figure 29) we have:

$$\xi = \frac{h\pi\pi}{h} = \frac{1}{\lambda}\left(1 - \sqrt{(1-\lambda)(1+\varphi)}\right),$$

$$\varphi = \frac{2\lambda}{h(1-\lambda)}\,\frac{t}{\chi}, \qquad t = \bar{S}_x\frac{\partial^2 F}{\partial y^2} + \bar{S}_y\frac{\partial^2 F}{\partial x^2} - 3\bar{X}_y\frac{\partial^2 F}{\partial x\, \partial y},$$

$$\chi = \bar{X}_x\frac{\partial^2 w}{\partial x^2} + \bar{Y}_y\frac{\partial^2 w}{\partial y^2} + 2\bar{X}_y\frac{\partial^2 w}{\partial x \partial y},$$

in which $\bar{X}_x \ldots \bar{S}_x$ are expressed by the initial stresses; that is, they are known as functions of the coordinates.

The approximate theory of stability agrees satisfactorily with the exact theory, as calculations of the plates show. It is based on the assumption that the variation of internal forces δT_1, δT_2, δS, equivalent to zero stresses on the contour of the plate always equal zero. This theory was discussed in Soviet and foreign literature and was recognized as the first completely fundamental theory of stability in plates and shells beyond the elastic limits. A systematic experimental verification of the approximate theory was conduced by us and abroad.

An approximate theory of stability in plates leads to a linear differential equation of deflection, which for constant initial stresses has the form:

$$\left(1-\frac{3}{4}\,\frac{1-\psi-K}{1-\psi}\,\bar{X}_x^2\right)\frac{\partial^4 w}{\partial x^4}+2\left(1-\frac{3}{4}\,\frac{1-\psi-K}{1-\psi}\,\bar{X}_x\bar{Y}_y\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+$$
$$+\left(1-\frac{3}{4}\,\frac{1-\psi-K}{1-\psi}\,\bar{Y}_y^2\right)\frac{\partial^4 w}{\partial y^4}=\frac{h\sigma_i}{(1-\psi)D}\left(\bar{X}_x\frac{\partial^2 w}{\partial x^2}+\bar{Y}_y\frac{\partial^2 w}{\partial y^2}\right).$$

Here k and ψ depend in a definite manner upon stress σ_i :

$$K=\frac{4\frac{d\sigma_i}{de_i}}{\left(\sqrt{E}+\sqrt{\frac{d\sigma_i}{de_i}}\right)^2},$$

$$\psi=\omega\left(1-\frac{1}{2}\sqrt{K}\right)\left[\left(1-\frac{1}{2}\sqrt{K}\right)^2+\frac{3}{4}\,\frac{K}{1-\left(1-\frac{3}{4}\sqrt{K}\right)\omega}\right]$$

The critical flexibility of the plate:

$$i=\frac{2c}{h}\sqrt{3(1-\nu^2)}\,.$$

can be found also from an equation similar to Timoshenko's equation:

$$i^2=\frac{E l^2\iint\left[(1-\psi)\left(\chi_1^2+\chi_1\chi_2+\chi_2+\chi_{12}^2\right)-\frac{3}{4}(1-\psi-K)\chi^2\right]dx\,dy}{-\iint\sigma_i\left[\bar{X}_x w_x^2+\bar{Y}_y w_y^2+2\bar{X}_y w_x w_y\right]dx\,dy}$$

Exact solutions are given for problems of uniformly compressed plates of arbitrary form in a plane and for certain rectangular plates. The approximations for a large number of problems are solved with the same degree of difficulty as in problems of elastic deformation. For example:

1. A bar and a narrow strip, given with freely drawn borders (Figure 30):

$$i=\frac{3l}{h}=\pi\sqrt{\frac{KE}{\sigma_i}};$$

2. A wide plate with two freely resting borders (Figure 31):

$$i=\frac{3l}{h}=\pi\sqrt{\frac{E(1-\psi+3K)}{4\sigma_i}}$$

3. A circular uniformly-compressed plate pinched on the periphery (Figure 32):

$$i=\frac{3R}{h}=3.84\sqrt{\frac{E(1-\psi+3K}{4\sigma_i}}$$

4. A long, narrow plate, freely resting along the whole periphery, compressed (contracted) in the direction of the long side (width b) (Figure 33):

$$i=\frac{3b}{h}=\pi\sqrt{\frac{2E(1-\psi)}{\sigma_i}\left[1+\sqrt{\frac{1-\psi+3K}{4(1-\psi)}}\right]};$$

5. A square, freely-resting plate of side a, compressed in one direction:

$$i=\pi\sqrt{\frac{E}{4\sigma_i}\left[13(1-\psi)+3K\right]},$$

6. A tube under the action of external pressure (Figure 34):

$$i = \frac{6\pi R}{h} = \pi\sqrt{\frac{3E}{\sigma_i}\left(1 - \psi + 3K\right)}. \qquad \sigma_i = p\frac{R}{h},$$

7. Lengthwise stability of a tube subjected to an axial force and side pressure (axial stress twice that of the tangential) (Figure 35):

$$i = \frac{3R}{h} = \frac{E}{\sigma_i}\sqrt{3K\left(1 - \omega + \frac{1}{2}\omega\sqrt{\lambda}\right)}.$$

Pressing Stamps

The plane problem of plasticity in a solid which does not possess hardening is interesting only as a problem of the supporting capacity of the solid. The basic plane problem of pressing stamps was solved by Prandtl. Figure 36 shows a picture of the lines of slipping during horizontal stamping. In the case "a," the pressure of cutting is

$$p = \tau_s(2 + \pi).$$

In the case "b":

$$p = 2\tau_s(1 + \gamma),$$

where τ_s is the limit of flow to shear: $\tau_s = \frac{\sigma_s}{\sqrt{3}}$

A uniform closed stress applied to the surface of a solid does not change the deviator of stress. Therefore, by converting Prandtl's problem (that is by applying a closed uniform tension (tensile force): $\sigma = 2\tau_s(1 + \gamma)$

and by changing the sign of the internal forces, we obtain a solution of the problem of pressing metal through a die (Figure 37) for the case where friction is absent. Pressure on wall AB, consequently, is equal to

$$p = 2\tau_s(1 + \gamma).$$

If boundary AA' is curved, then polar coordinates will yield another solution. The pressure on the wall then will be

$$p = 2\tau_s\left(1 + \ln\frac{r}{a}\right).$$

For a curvilinear stamp of radius R the following relation holds between force P, radius of imprint a, and its depth of penetration δ :

$$P = 4\tau_s a\left(1 + \frac{\pi}{2} - \sqrt{\frac{\delta}{2R}}\right).$$

The axial-symmetrical problem (as opposed to the plane problem) of stamp pressing was solved by A. Yu. Ishlinskiy. V. M. Puchkov showed that in this case it is possible to accept the same lines of slipping as in the corresponding plane problem and gave a formula that related force of pressure P, area F, and volume V of the imprint (Figure 38) with each other:

$$P = p_A F + \frac{p_0 - p_A}{\delta}V.$$

In this expression we have:

$$p_A = \sigma_s(1 + \gamma_A), \qquad p_0 = p_s\left(1 + \gamma_0 + \frac{1}{2}\int_0^B \frac{dr - dz}{r}\right),$$

in which the integral is taken along the curve BO.

Dynamic Problems

The stating and solving of the basic unidimensional dynamic problems of plasticity belongs to Rakhmatulin. If instantaneous pressure impulse is applied to the end of a rod, then a group of compression waves will travel lengthwise along the rod.

The first waves of this group will travel as elastic waves of velocity $a_c = \sqrt{\frac{E}{\rho}}$; then comes a "packet" of plastic waves of variable speed (Figure 39) $a = \sqrt{\frac{1}{\rho_0}\frac{d\sigma}{d\varepsilon}}$; and finally a "load-relieving" wave (Figure 40).

After the passing of the "load-relieving" waves, the rod will have a variable limit of flow, insofar as for every section there is a point A from which "load-relieving" begins.

The mathematical problem leads to the solution of a complex-function equation of a hyperbolic type; that is, to a new problem in mathematical physics. Kh. A. Rakhmatulin's solution was extended to formulas for any arbitrary law describing the falling off of pressure at the end of the rod. Kh. A. Rakhmatulin's second important problem relates to the problem of transverse blows across a wire or rod, in which plastic deformations also are assumed to arise.

The nature of the wave process is shown in Figure 41. A certain critical speed of the blow on the wire was found which causes the destruction of the wire. Tests with rubber braids confirm the authenticity of the wave picture as given by Rakhmatulin.

The polar-symmetrical dynamic problem was examined by L. V. Al'tshuler, Kh. A. Rakhmatulin, and F. A. Bakhshiyan.

/Appended figures follow./

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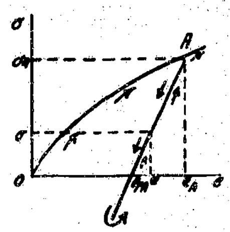

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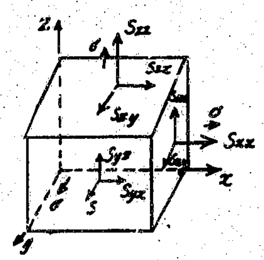

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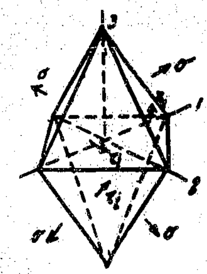

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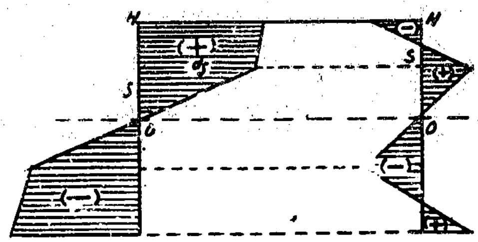

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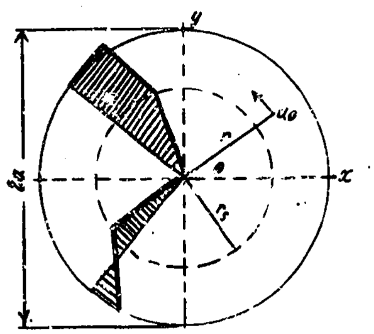

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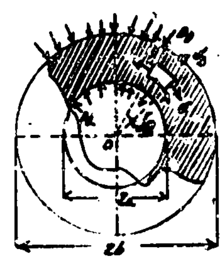

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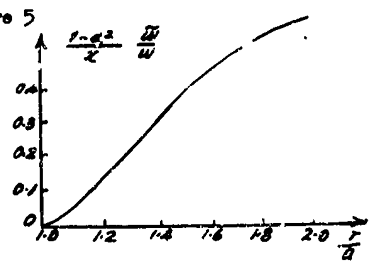

Figure 7

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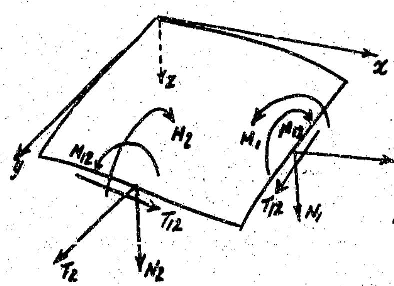
Figure 8



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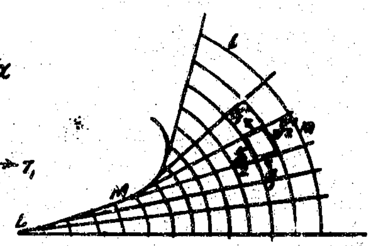


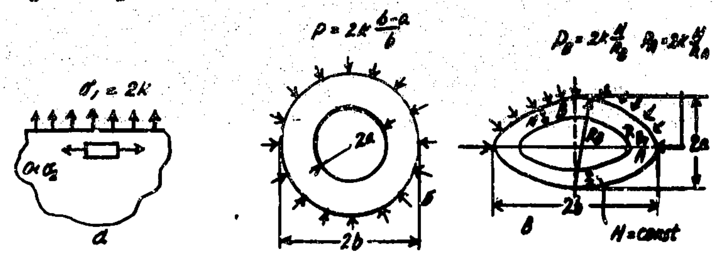
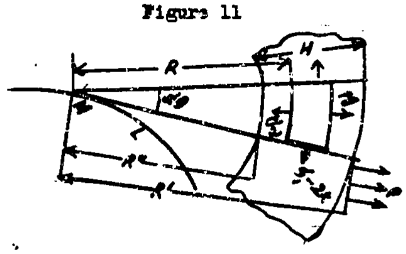


Figure 10

Figure 11



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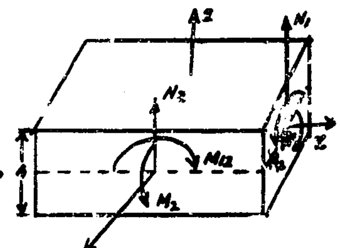

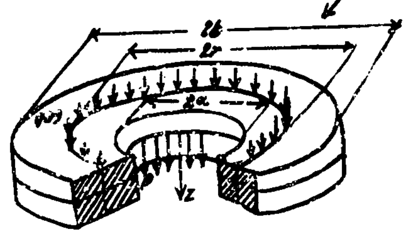

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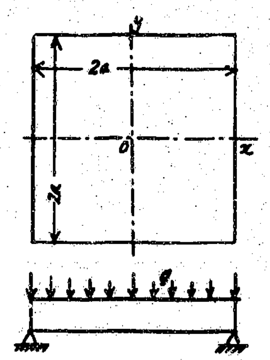

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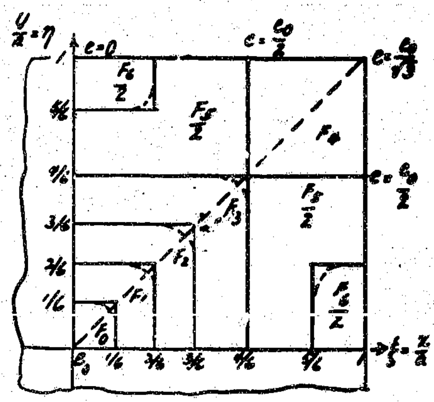

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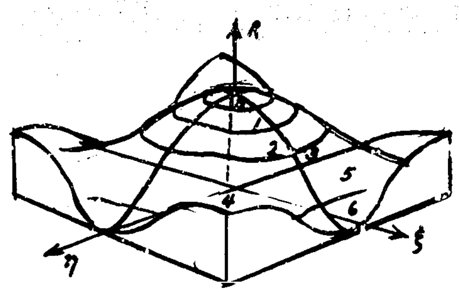

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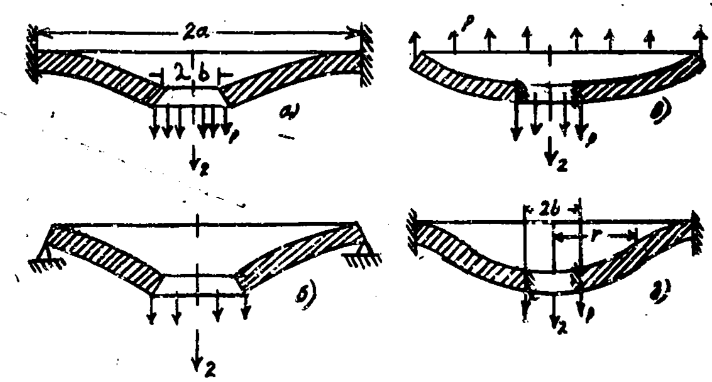

Figure 17

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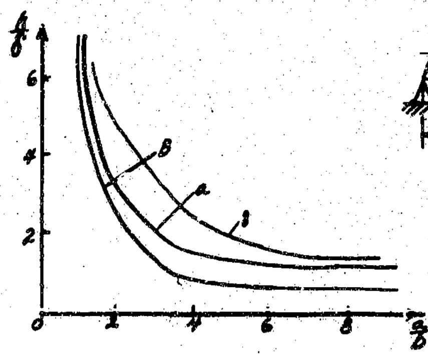
Figure 18



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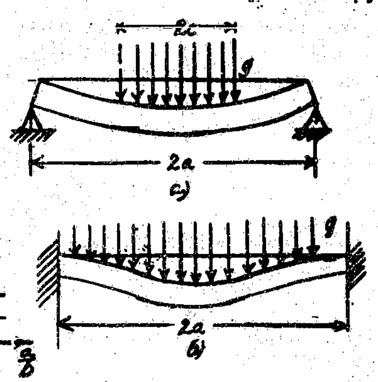

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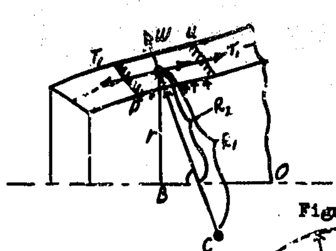

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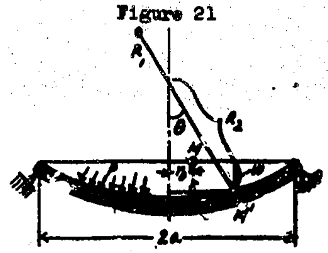

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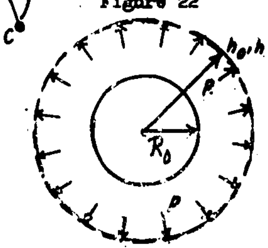

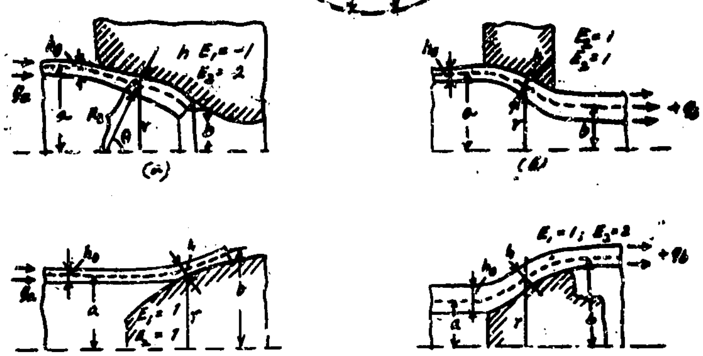

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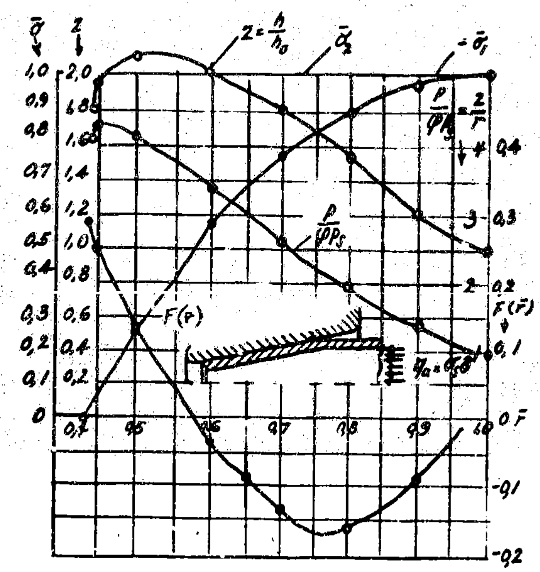

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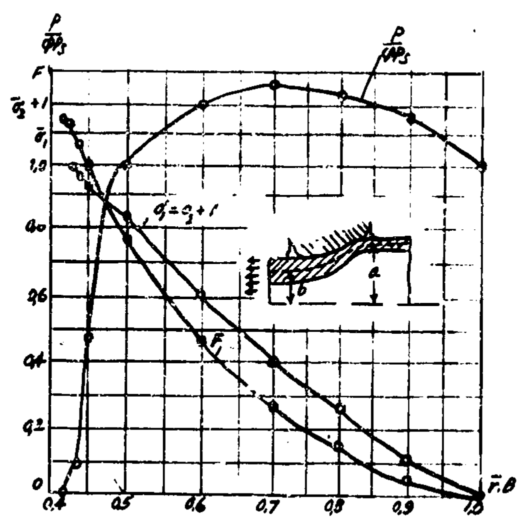

Figure 25

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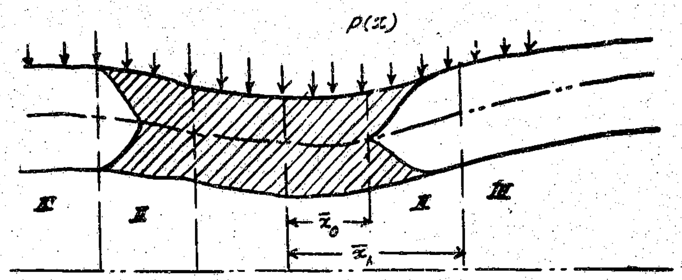

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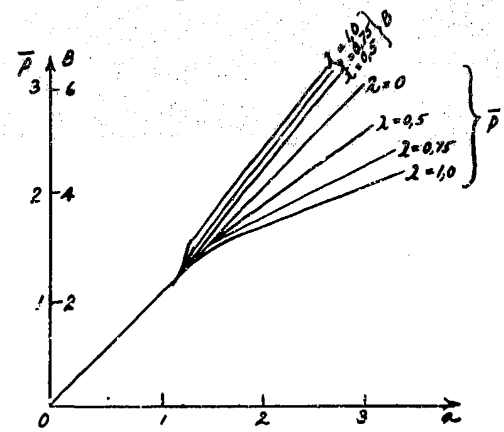

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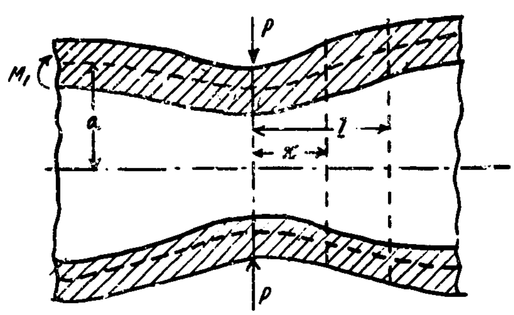

Figure 28

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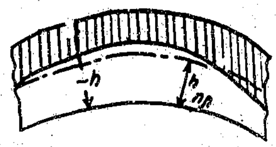

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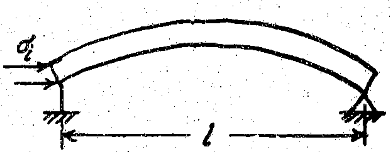

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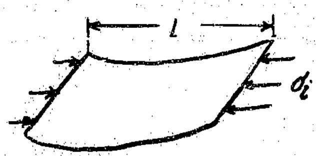

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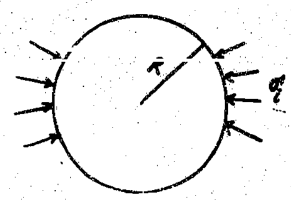

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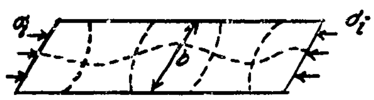

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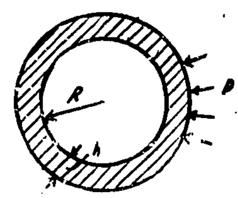

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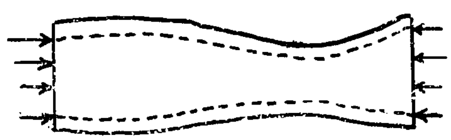

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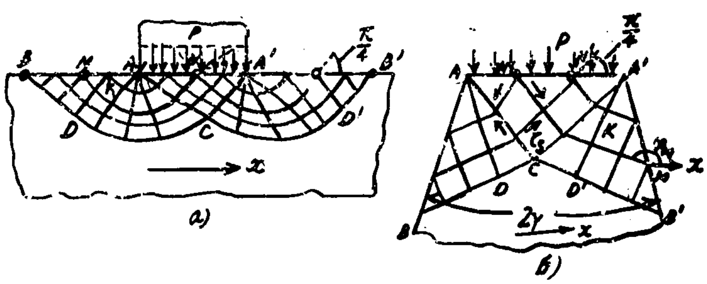

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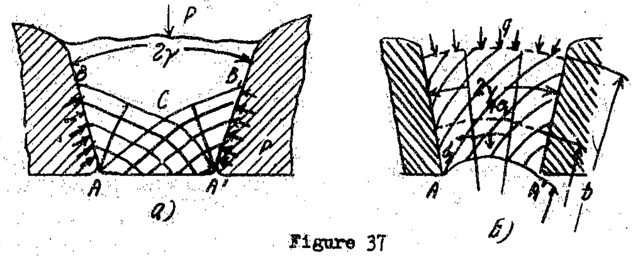

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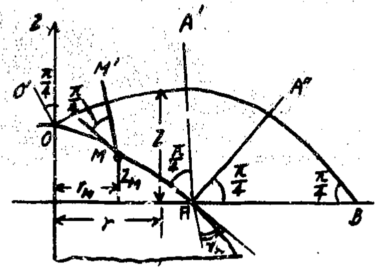

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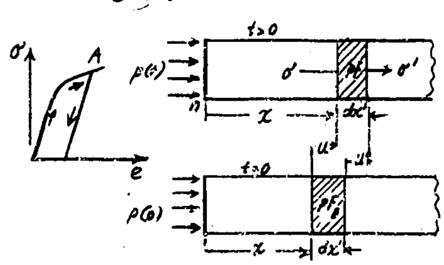

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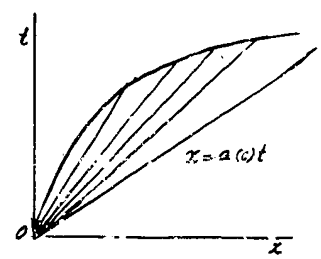

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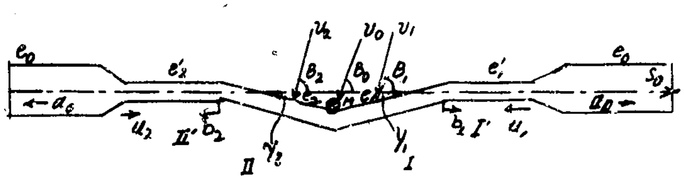

Figure 41

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